

Properties of the Skeleton of Aggregates Grown on a Cayley Tree

Shlomo Havlin,^{1,2} James E. Kiefer,¹ George H. Weiss,¹ Daniel Ben-Avraham,² and Yehoshua Glazer²

Received February 26, 1985; revision received June 11, 1985

We discuss and analyze a family of trees grown on a Cayley tree, that allows for a variable exponent in the expression for the mass as a function of chemical distance, $\langle M(l) \rangle \sim l^d$. For the suggested model, the corresponding exponent for the mass of the skeleton, d_s^i , can be expressed in terms of d_i as $d_s^i = 1$, $d_i \leq d_i^c = 2$; $d_s^i = d_i - 1$, $d_i \geq d_i^c = 2$, which implies that the tree is finitely ramified for $d_i \leq 2$ and infinitely ramified when $d_i \geq 2$. Our results are derived using a recursion relation that takes advantage of the one-dimensional nature of the problem. We also present results for the diffusion exponents and probability of return to the origin of a random walk on these trees.

KEY WORDS: Fractals; Cayley trees; chemical distance; diffusion on trees.

1. INTRODUCTION

Two of the intrinsic properties that can be used to characterize random aggregates^(1,8) are the total mass, $\langle M \rangle$, and mass of the skeleton, $\langle M_s \rangle$, where the skeleton of a random aggregate is defined⁽⁸⁾ to be the totality of all shortest (or chemical) paths along the aggregate between two specified subsets. For structures grown on the Cayley tree it is convenient to define the subsets to consist, respectively, of a point (the origin) and a surrounding shell. Dead ends not reaching the shell do not belong to the skeleton. On any aggregates one can consider the two masses mentioned earlier, as a function of the chemical distance, l , along the skeleton where the chemical distance between two points is defined as the shortest distance

¹ National Institutes of Health, Bethesda, Maryland 20205.

² Department of Physics, Bar-Ilan University, 52100 Ramat Gan, Israel.

between them as measured along the cluster. These relations can often be characterized by scaling laws, so that

$$\langle M(l) \rangle \sim l^{d_l}, \quad \langle M_s(l) \rangle \sim l^{d_s^l} \quad (1)$$

which defines the two exponents d_l and d_s^l . In this paper we discuss a new family of models for the growth of random aggregates on a Cayley tree that allows for arbitrary values of d_l and d_s^l . The origin of the resulting aggregate is a point of asymmetry, thus we term such aggregates "point fractals." Because the aggregate is generated sequentially one can solve the equations describing the growth recursively so that no simulations are required in the analysis.

2. DEFINITION OF THE MODEL

The aggregate to be analyzed is one grown on a Cayley tree characterized by discrete levels, grown from a point chosen to be the origin. In our model, each node on a given level is allowed to grow to i nodes on the succeeding level, where $0 \leq i \leq n$, where i is chosen probabilistically by a rule to be discussed below. The integer n is the maximum allowable number of daughters that stem from a single node. The rule for choosing i will insure that the average number of nodes in the structure $\langle M(l) \rangle$ varies as

$$\langle M(l) \rangle \sim l^{d_l} \quad (2)$$

for $l \gg 1$, where d_l can be varied. Let $g(l)$ be the average number of nodes (grown from a single node) in level l and let $\langle B(l) \rangle$ be the average number of nodes in l . This quantity is related to $g(l)$ by

$$\langle B(l) \rangle = \prod_{l'=1}^{l-1} g(l') \quad (3)$$

We will choose the form of $g(l)$ so that Eq. (1) is satisfied. If we assume the form

$$g(l) = 1 + \alpha/l \quad (4)$$

where α is a constant to be chosen to insure the validity of Eq. (2), then

$$\ln \langle B(l) \rangle = \sum_{l'=1}^{l-1} \ln \left(1 + \frac{\alpha}{l'} \right) \sim \alpha \sum_{l'=1}^{l-1} \frac{1}{l'} \sim \alpha \ln l \quad (5)$$

or

$$\langle B(l) \rangle \sim l^\alpha \quad (6)$$

The expected total number of nodes in the aggregate is therefore

$$\langle M(l) \rangle \sim \sum_{l'=1}^l \langle B(l') \rangle \sim l^{\alpha+1}/(\alpha+1) \tag{7}$$

allowing us to make the identification

$$d_l = \alpha + 1 \tag{8}$$

for this model. Notice that since $g(l)$ is an average value there are an infinite number of ways of choosing probabilities to assure the validity of Eq. (4). For example, if a single node on level l can grow to at most two nodes on level $l+1$, then we can choose for $P(i)$, the probability that i new nodes are grown ($i \leq 2$):

$$P_l(0) = \frac{1}{4} - \frac{2\alpha}{3l}, \quad P_l(1) = \frac{1}{2} + \frac{\alpha}{3l}, \quad P_l(2) = \frac{1}{4} + \frac{\alpha}{3l} \tag{9}$$

Since the relations in Eqs. (2) and (6) are asymptotic, the parameter α can be chosen arbitrarily since Eq. (4) need only hold for $l \gg 1$. The probabilities in Eq. (9) generalize the usual construction⁽⁶⁾ of a percolation cluster on the standard Cayley tree which corresponds to $\alpha = 0$.

Among the various possibilities we choose to study the mean field of the tree growth model (TGM) for trees grown on regular lattices.⁽⁹⁾ In the present TGM exactly $[l^\alpha]$ nodes are added at level l , where “[]” means “largest integer contained in.” These are chosen at random from the $n[(l-1)^\alpha]$ possibilities allowed by the nodes in level $l-1$, where n is the maximum number of nodes generated from a single one. When $n=2$, the retention of lowest-order terms in l^{-1} and $l^{-\alpha}$ for large l , leads to the result

$$P_l(0) \sim \frac{1}{4} \left(1 - \frac{2\alpha}{l} \right) - \frac{1}{8l^\alpha}, \quad P_l(1) \sim \frac{1}{2} \left(1 + \frac{1}{2l^\alpha} \right) \tag{10}$$

$$P_l(2) \sim \frac{1}{4} \left(1 + \frac{2\alpha}{l} \right) - \frac{1}{8l^\alpha}$$

Notice that in the present model there are no terminating clusters.

3. STATISTICAL PROPERTIES OF THE SKELETON

So far we have presented results dealing with the expected number of nodes contained in an aggregate containing l levels. We next consider some of the properties of the skeleton of the TGM defined above. To calculate

the behavior of $\langle B^s(l) \rangle$, the average number of nodes in the skeleton, we make use of a recursion relation valid for $l \gg 1$, that essentially neglects the fact that l is an integer. An exact combinational recursion relationship outlined in the Appendix can also be derived. This relation was found to be very time consuming to evaluate numerically, which led us to develop an alternate approximation. Several values generated by the approximate recursion relation were checked by using the exact one and found to be in excellent agreement with them; hence we restrict this discussion solely to the asymptotic recursion and consider the case in which a single node can grow to a maximum of two nodes in the following level. The generalization to a larger number of nodes is trivial. The quantity of interest in our analysis will be denoted by a_l , which is defined to be

$$a_l = \langle B^s(l) \rangle / \langle B(l) \rangle \quad (11)$$

i.e., the fraction of the number of nodes in the l th level that belong to the skeleton. The definition of our model implies that the quantity

$$\theta_l = \langle B(l) \rangle / [2 \langle B(l-1) \rangle] = l^\alpha / [2(l-1)^\alpha] \quad (12)$$

is the probability that a node at level $l-1$ will produce a connected node at level l . Finally we let q_l be the conditional probability that a node at level $l-1$ that produces at least one node at level l , will produce exactly two nodes. Thus q_l can be expressed in terms of θ_l as

$$q_l = \theta_l^2 / [\theta_l^2 + 2\theta_l(1 - \theta_l)] = \theta_l / (2 - \theta_l) \quad (13)$$

With these definitions in hand we can write the following recursion relation for the a_l :

$$\begin{aligned} a_{l-1} &= [1 - (1 - \theta_l)^2] \{ [1 - (1 - a_l)^2] q_l + a_l(1 - q_l) \} \\ &= (2 - \theta_l a_l) \theta_l a_l \end{aligned} \quad (14)$$

The first term in the square brackets represents the probability that a node at level $l-1$ will produce at least one node at level l . The second factor, in curly brackets, gives the contributions from double bonded nodes (with probability q_l) and those from singly bonded nodes. Equation (14) is a backwards recursion which is to be started from a level $L \gg 1$. Since any node that appears at this final level is necessarily a part of the skeleton one starts the recursion with $a_L = 1$, together with the values of θ_l shown in Eq. (12). The results of using the recurrence Eq. (14) are shown Fig. 1 as a plot of $\ln \langle B^s(l) \rangle$ as a function of $\ln l$, for different values of α and L . This suggests that as L increases $\ln \langle B^s(l) \rangle / \ln l \equiv A_l$ approaches a constant for

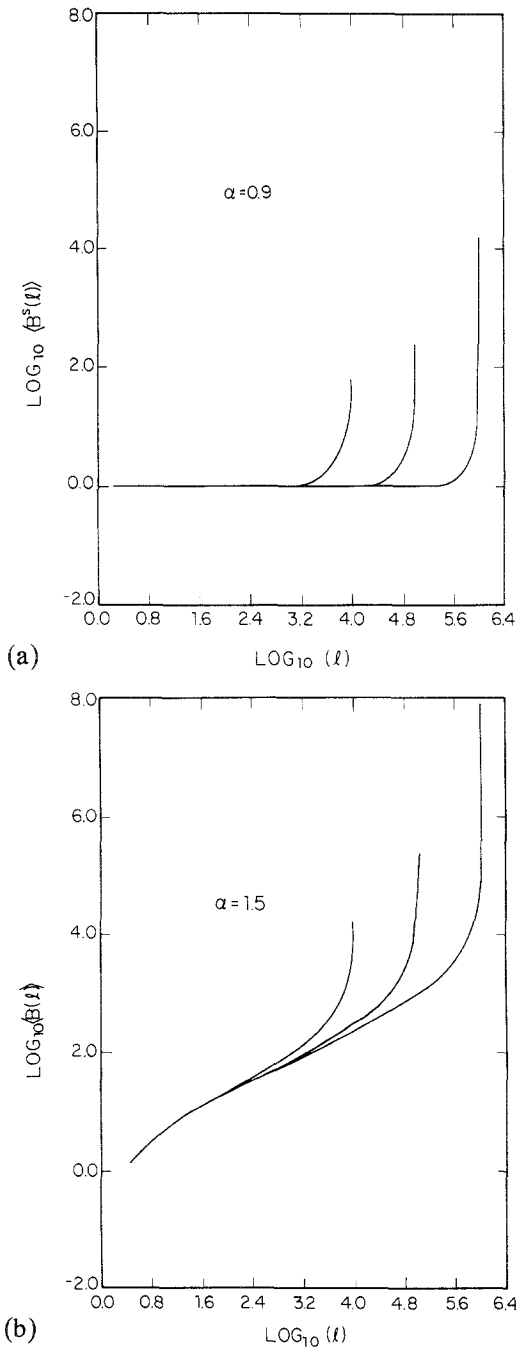


Fig. 1. Plot of $\log_{10}\langle B^s(l) \rangle$ vs. $\log_{10} l$ for $L = 10^4, 10^5,$ and 10^6 for (a) $\alpha = 0.9$ (b) $\alpha = 1.5$.

$1 \ll l \ll L$, the constant depending on whether α is greater, or less than 1. For any $\alpha < 1$, A_l approaches zero as $l \rightarrow \infty$ and for $\alpha > 1$, A_l approaches the value $\alpha - 1$. This behavior can be verified analytically from the recurrence relation in Eq. (14). Let suppose that $\langle B^s(l) \rangle$ has the asymptotic scaling form

$$\langle B^s(l) \rangle \sim K \cdot l^{\alpha_s}, \quad l \gg 1 \quad (15)$$

where K is a constant. Then, because of Eq. (11) it follows that

$$a_l \sim K' l^{\alpha_s - \alpha} \quad (16)$$

where K' is another constant. If this, together with the large l form for θ_l ,

$$\theta_l \sim \frac{1}{2} \left(1 + \frac{\alpha}{l} \right) \quad (17)$$

is substituted into Eq. (14), and only the lowest-order terms in l are retained, then one finds

$$\alpha_s = \frac{K'}{4} l^{\alpha_s - (\alpha - 1)} \quad (18)$$

In order for this relation to be consistent as $l \rightarrow \infty$ it is necessary that $\alpha_s = \alpha - 1$ whenever $\alpha \geq 1$, so that $K' = 4(\alpha - 1)$. When $\alpha < 1$, since it is assumed that $\alpha_s > 0$, it follows from the fact that $\alpha_s - (\alpha - 1) > 0$ that $K' = 0$ which implies also that $\alpha_s = 0$. The relation between d_l and α is given in Eq. (8), and similarly d_l^s and α_s are related by $d_l^s = \alpha_s + 1$. Consequently the facts just established for the α 's imply that

$$\begin{aligned} d_l^s &= d_l - 1, & d_l &\geq d_l^s = 2 \\ &= 1, & d_l &\leq d_l^s \end{aligned} \quad (19)$$

In a recent study⁽¹⁰⁾ we have found that the chemical diffusion exponent, d_w^l , defined by the relation

$$\langle l \rangle^{d_w^l} \sim t \quad (20)$$

can be expressed in terms of d_l and d_l^s by

$$d_w^l = 2 + d_l - d_l^s \quad (21)$$

In Eq. (20) $\langle l \rangle$ is the average displacement of a random walk that starts at

the origin of the tree. The combination of Eq. (19) and the results just given leads to

$$\begin{aligned} d_w^l &= d_l + 1, & d_l \leq 2 \\ &= 3, & d_l \geq 2 \end{aligned} \tag{22}$$

The result for $d_l = 2$ is the same as found for diffusion on percolation clusters on a Cayley tree.⁽²⁾

4. SUMMARY AND DISCUSSION

To summarize, we have presented a model of random aggregates grown on the Cayley tree in which the intrinsic dimension, d_l , of these aggregates can be varied. The model is a mean-field analog of the corresponding aggregates grown in regular space.⁽⁹⁾ We find that for d_l below $d_l^c = 2$ the aggregate's skeleton is linear, while for $d_l > d_l^c = 2$ the skeleton's mass scales with l as $\langle M \rangle_s \sim l^{d_l^s}$, where $d_l^s = d_l - 1$. When diffusion only occurs on the skeleton $d_w^l = 2$ for all d_l as shown elsewhere.⁽⁹⁾ We see that the effect of dead ends is to increase d_w^l by 1 (from 2 to 3) for $d_l \geq 2$. It is also interesting that although the exponent d_w^l is independent of d_l when $d_l \geq 2$ the probability of return to origin, characterized by the fracton dimensionality \bar{d} , does vary with d_l .⁽⁷⁾

$$\begin{aligned} \bar{d} &= 2d_l / (d_l + 1), & d_l \leq 2 \\ \bar{d} &= 2d_l / d_w^l = 2d_l / 3, & d_l \geq 2 \end{aligned} \tag{23}$$

The model studied in this paper can be regarded as a mean-field analog of the TGM⁽⁹⁾ consisting of random treelike aggregates grown on regular lattices. Numerical results for the TGM on a square lattice shows that for $d_l < d_l^c = 1.65 \pm 0.05$ the resulting skeleton is linear ($d_l^s = 1$), while for $d_l > d_l^c$ it is nonlinear ($d_l^s > 1$). This bifurcation in behavior is also reproduced in the present model. Finally, we note that if $\bar{\zeta}_l$ is the exponent of the total resistance between the origin and level l , i.e., $R(l) \sim l^{\bar{\zeta}_l}$ for large l , then $\bar{\zeta}_l$ satisfies⁽¹⁰⁾

$$\bar{\zeta}_l = d_w^l - d_l \tag{24}$$

which implies by Eq. (22), that $\bar{\zeta}_l = 1$ for $d_l \leq 2$ and $\bar{\zeta}_l = 2 - d_l^s = 3 - d_l$ for $d_l \geq 2$. This also agrees with the relation found by a scaling argument for the resistance between the origin and shell l of the tree.⁽¹⁰⁾

APPENDIX

To calculate d_l^s we must find an expression for $\langle B^s(l) \rangle$. This quantity can be written as an average over the probability that level l has i nodes, $P_l(i)$:

$$\langle B^s(l) \rangle = \sum_{i=1}^{B(l)} i P_l(i) \quad (\text{A1})$$

The probabilities $\{P_l(i)\}$ can be derived by means of a recursion relation. Let $f(i, k, n)$ be the probability that exactly i nodes will have one or two growing branches conditional on k nodes at a given level growing to n nodes on the following level. Then we have

$$P_{l-1}(i) = \sum_{j=1}^{2i} P_l(j) f(i, B(l-1), j) \quad (\text{A2})$$

together with a starting value $P_L(i) = \delta_{B(L), i}$. This starting value is valid for the present model in which the $B(l)$'s are deterministic. The case of random $B(l)$ will be considered in a future investigation.

Finally, in order to use Eqs. (A1) and (A2) we need an expression for $f(i, k, n)$. This can be derived by a combinatorial argument and is found to be

$$f(i, k, n) = \frac{k! 2^{2i-n}}{(n-i)!(2i-n)!(k-i)!} \bigg/ \frac{(2k)!}{n!(2k-n)!}, \quad 0 \leq i \leq n \leq 2i \leq 2k \quad (\text{A3})$$

where $n-i$ is the number of sites having both nodes grown, $2i-n$ have one node grown each and $k-i$ having no nodes grown.

REFERENCES

1. B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
2. S. Havlin, in *Proceedings of the International Conference on the Kinetics of Aggregation and Gelation*, F. Family and D. Landau, eds. (North-Holland, Amsterdam, 1984); S. Havlin and R. Nossal, *J. Phys.* **A17**:L427 (1984).
3. H. J. Herrman, D. C. Hong, and H. E. Stanley, *J. Phys.* **A17**:L261 (1984).
4. A. L. Ritzenberg and R. J. Cohen, *Phys. Rev. B* **30**:2120 (1984).
5. J. Vannimenus, J. P. Nadal, and H. Martin, *J. Phys.* **A17**:L351 (1984).
6. Z. Alexandrowicz, *Phys. Lett.* **80A**:284 (1980).
7. S. Havlin, Z. V. Dzordjevic, I. Majid, H. E. Stanley, and G. H. Weiss, *Phys. Rev. Lett.* **53**:178 (1984).
8. S. Havlin, R. Nossal, B. Trus, and G. H. Weiss, *J. Phys. A. Math. Gen.* **17**:L957 (1984).
9. S. Havlin, R. Nossal, and B. Trus, preprint.
10. S. Havlin, R. Nossal, B. Trus, and G. H. Weiss, *Phys. Rev. B* **31**:7497 (1985).